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Modulation of the amplitude of steep wind waves

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Lake & Yuen (1978) have suggested that in very steep wind waves the modulation-frequency of the wave amplitude may correspond to the frequency of the fastest-growing subharmonic instability of a uniform train of waves whose amplitude equals the mean wave amplitude $\bar{a}$. The approximate theory of Benjamin & Feir (1967) gives this frequency as $(\bar{a}k)f_d$, where k is the wavenumber and f_d the frequency of the unperturbed waves. This expression applies strictly only to very small values of the wave steepness $\bar{a}k$.

More recently (Longuet-Higgins 1978) the present author calculated accurately all the normal-mode instabilities of steep gravity waves on deep water. In this note these calculations are used to determine the frequency of the fastest-growing subharmonic instabilities precisely. When compared with the experimental data of Lake & Huen, these frequencies show even closer agreement.

1. Introduction

In the study of random seas, questions concerning the fluctuation in the height of the waves – for example, how many waves, on the average, are there in a wave group? – have important applications to naval architecture and ocean engineering, especially when we are concerned with nonlinear phenomena (such as ship slamming) stimulated by a resonant response to the waves.

In the past such questions have been treated mainly by linear theories (Longuet-Higgins 1956; Goda 1970; Ewing 1973). Thus, there is a demonstratable relation between the group-length of a gaussian stochastic process and the 'width' of the corresponding frequency spectrum, suitably defined. Such linear theories do not predict any particular value for the width of the spectrum, but leave this as a matter either for measurement or to be estimated by wave-forecasting techniques.

In very steep waves, however, the waves become non-gaussian, and the frequency spectrum, as ordinarily defined, becomes contaminated by the presence of second and higher harmonics which are phase-bound to the corresponding fundamental Fourier components. So the appropriate width of the spectrum tends to be over-estimated, when spectral moments are used. Against this, it has appeared that the width of the main peak in the spectrum, under conditions of active wave generation, can be extremely narrow (Hasselmann *et al.* 1973).

Now it is known (Benjamin & Feir 1967) that a uniform train of gravity waves of finite amplitude a in deep water is inherently unstable to certain subharmonic perturbations. Benjamin & Feir represented these as side-bands, with radian frequencies $\sigma \pm \Delta\sigma$, where σ is the frequency of the fundamental and $\Delta\sigma$ a positive perturbation;

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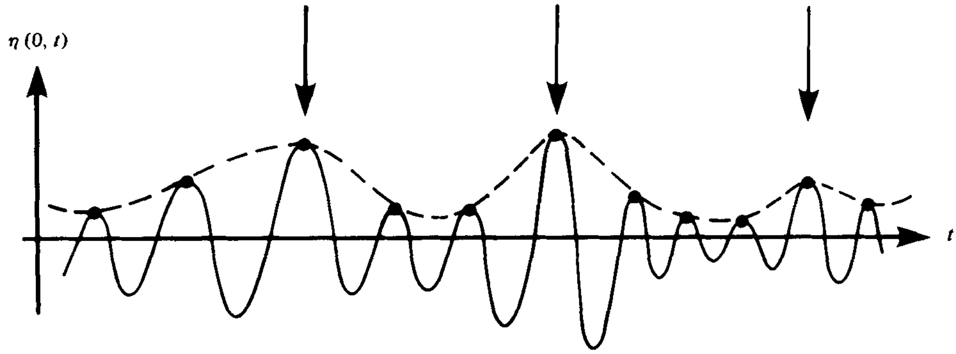

FIGURE 1. Sketch of the surface elevation η at a fixed point, as a function of the time t.

and they showed that the side-bands would grow at the expense of the fundamental wave provided

$$\Delta\sigma/\sigma < \surd 2\, ak, \tag{1.1}$$

k being the fundamental wavenumber. Moreover, the most rapidly growing perturbation (of this type) has a frequency such that

$$\Delta\sigma/\sigma = ak. \tag{1.2}$$

This interesting result was taken up by Lake & Yuen (1978) who suggested that the fluctuations of the wave envelope in a steep, *irregular* wave train might correspond to the most rapidly growing perturbations of a *uniform* wave train. In support of this not unreasonable suggestion they measured the ratio of the modulation frequency f_e to the dominant wave frequency f_d of wind-waves in a short channel, with fetches up to 30 ft and wind speeds up to 30 ft s^{-1}. Despite some scatter, f_e/f_d appeared to increase about linearly with mean wave steepness (see Lake & Yuen (1978) figure 13; also see figure 6 below).

Now the theoretical results (1.1) and (1.2) are valid strictly only for small but finite values of $\Delta\sigma$ and ak. Recently the present author has calculated accurately the instabilities of deep-water gravity waves over the greater part of the range of wave steepness ak (Longuet-Higgins 1978). These calculations have been confirmed by an entirely different method (Longuet-Higgins & Cokelet 1978). The results show that the instabilities are of two distinct types: local instabilities, which lead directly to whitecapping, and subharmonic instabilities of the Benjamin–Feir type, which however are confined to the range $0 < ak < 0{\cdot}36$ approximately. It was shown that the relations (1.1) and (1.2) are valid only at the lower end of the range of ak.

In this note it will be shown that the results of Longuet-Higgins (1978) allow the frequency of the most rapidly growing subharmonic perturbation to be determined accurately. When the resulting curve of f_e/f_d is substituted for the straight line (1.2) used by Lake & Yuen, rather better agreement with their data is obtained.

2. The modulation frequency

Consider a record $\eta(t)$ of the sea surface elevation at some fixed horizontal position, say $x = 0$, as shown schematically in figure 1. The broken line represents the modulus

$\eta_e = |A(t)|$ of the envelope function, drawn so as to touch the dominant waves near to their crests. The envelope is in fact the smoothest such function (in a certain sense) which can be so drawn. We wish to find the mean frequency (in some sense) of the fluctuations of $\eta_e(t)$.

Now in the proposed model we regard $\eta(t)$ as the result of modulating a *uniform* train of waves of finite amplitude, that is an unperturbed wave

$$\bar{\eta} = f(kx-\sigma t) = \sum_j a_j \cos j(kx-\sigma t) \tag{2.1}$$

with a_j real. Here x and t represent the horizontal co-ordinate and the time. To retrieve $\bar{\eta}(t)$ we set $x = 0$.

It is useful to consider the perturbations of $\bar{\eta}$ in a frame of reference *moving with the phase-speed* $c = \sigma/k$. Then $\bar{\eta}$ itself becomes independent of the time:

$$\bar{\eta} = f(kx') = \sum_j a_j \cos jkx', \tag{2.2}$$

where $x' = x - ct$. It is further convenient to express $\bar{\eta}$ as a function of the velocity potential ϕ in this relative motion, so that

$$\bar{\eta} = F(\phi) = \sum_j A_j \cos(jk\phi/c). \tag{2.3}$$

At a wave crest we take $x' = 0$, $\phi = 0$ so the wave *amplitude* a is given by

$$a = \sum_j a_j \div \sum_j A_j. \tag{2.4}$$

We now may express the elevation η in the form

$$\eta = \bar{\eta}(\phi) + \eta'(\phi, t), \tag{2.5}$$

where η' is a small perturbation of the basic wave $\bar{\eta}$. Squares and higher powers of η' (but not $\bar{\eta}$) are neglected. It is relatively straightforward to calculate the *normal mode perturbations*, which take the form

$$\eta'_n = [P_n(\phi) + iQ_n(\phi)]\, e^{-i\sigma_n t}, \tag{2.6}$$

where P_n and Q_n are real and imaginary parts of the complex amplitude function: n is a rational number characteristic of the particular normal mode; and

$$\sigma_n = \alpha_n + i\beta_n \tag{2.7}$$

say. The real part α_n gives the radian frequency of the normal mode, in the travelling reference frame, and the imaginary part β_n gives its e-folding rate of growth.

The normal modes were calculated explicitly in Longuet-Higgins (1978). It was shown that in general we can write

$$n = l/m, \tag{2.8}$$

where l and m are integers, m denoting the number of wavelengths (in a horizontal direction) over which the particular mode is periodic. Thus $m = 1$, $l \geqslant 1$ corresponds to all the *superharmonics* having length scales equal to or less than the wavelength $2\pi/k$, while $m = 2$, $l = 2$, for example, corresponds to all the subharmonics which are *two* wavelengths long; and so on.

The values of l are assigned to each harmonic in such a way that the ratio $n = l/m$ gives the number of maxima and minima of each harmonic *at low values of ak*; positive

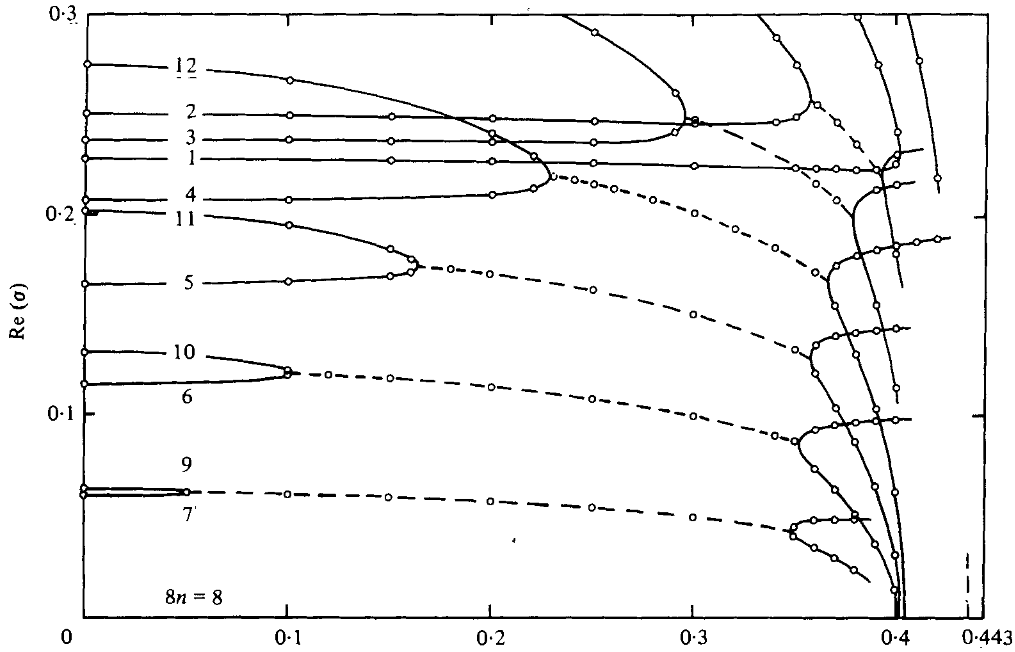

FIGURE 2. Real part of the radian frequency σ of normal-mode perturbations of deep-water gravity waves, as a function of the wave amplitude ak (after Longuet-Higgins 1978).

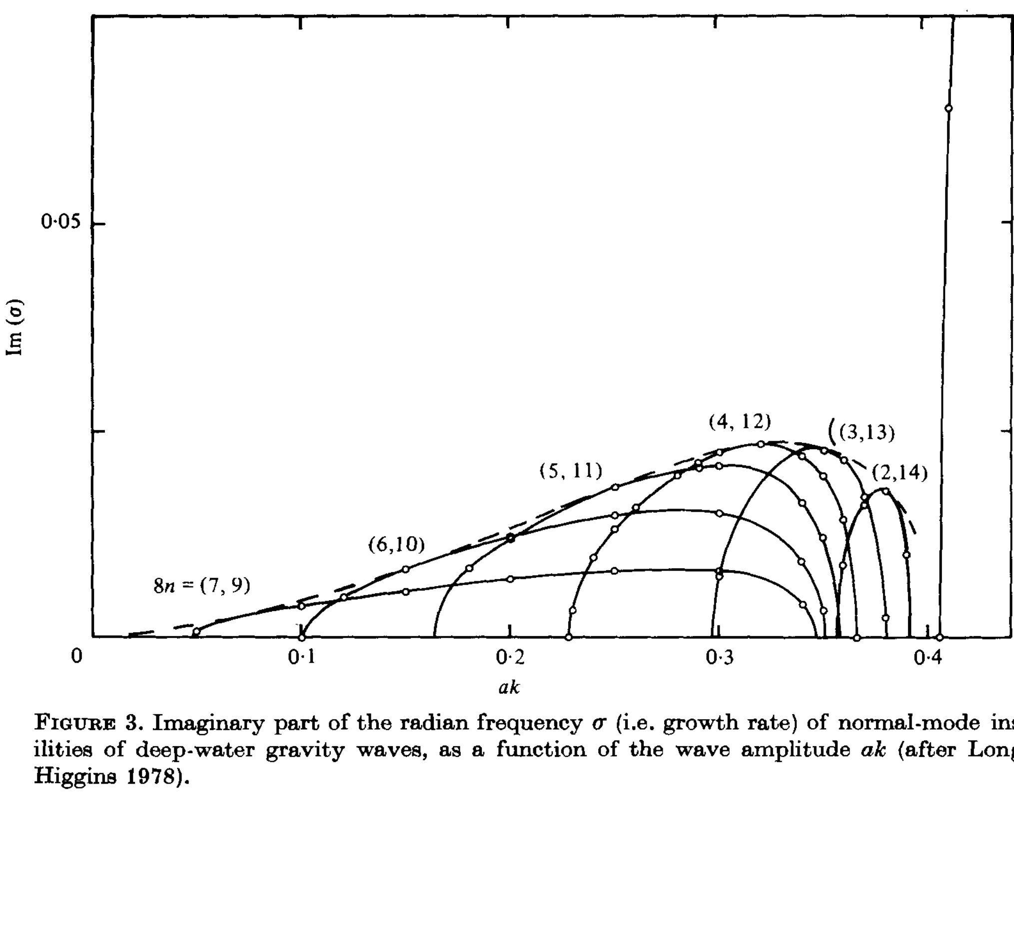

FIGURE 3. Imaginary part of the radian frequency σ (i.e. growth rate) of normal-mode instabilities of deep-water gravity waves, as a function of the wave amplitude ak (after Longuet-Higgins 1978).

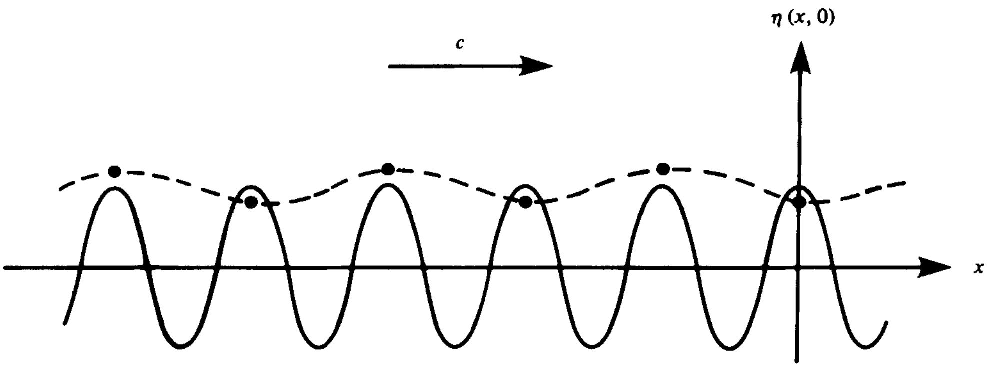

FIGURE 4. Sketch of the surface elevation η of a perturbed train of waves as a function of the horizontal distance x.

values of n correspond to modes propagated in the same sense as the unperturbed wave, negative values of n to modes propagated in the opposite sense.

Some of the modes were connected analytically to more than one mode at low ak. For these it is necessary to define a *pair* (or more) of wavenumbers, as in figure 3 below. Figures 2 and 3 respectively show the real and imaginary parts of σ when $m = 8$, so $l = 8n$.

Now we need to obtain the envelope function $\eta_e(t)$ in the *stationary* frame of reference. To do this we sample $\eta(t)$ at points near the crests $\phi = NcL$ and at times $t = N\tau$, where N is an integer and τ is the basic wave period,

$$L = 2\pi/k, \quad \tau = 2\pi/\sigma_d. \tag{2.9}$$

Thus we consider the set of points

$$\eta(N\tau) = a + [P(NcL) + iQ(NcL)]\, e^{-i\sigma_n t}, \tag{2.10}$$

where $t = N\tau$, as defining the wave envelope η_e approximately (see figure 4).

For example, when $m = 2$ and we consider odd harmonics (l odd), P and Q will reverse sign when ϕ is increased by cL (t being fixed). In other words, on spatially adjacent waves the perturbations are of opposite sign. Hence, as each crest passes the fixed point of observation ($x = 0$) the vector $[P(NcL) + iQ(NcL)]$ will be constant in magnitude but will alternate in sign; hence it will oscillate with radian frequency $\frac{1}{2}\sigma_d$. Meanwhile the vector $e^{-i\sigma_n t}$ will rotate with radian frequency α_n. So the apparent frequency, in the stationary frame of reference, is $(\frac{1}{2}\sigma_d - \alpha_n)$. The term $\frac{1}{2}\sigma_d$ clearly represents a Doppler shift, compounded of the phase-velocity and the wavenumber $\frac{1}{2}$ of the basic disturbance.

More generally, when the normal mode has two dominant components with wavenumbers n_1 and n_2, say, the dominant spatial wavenumber will be $\Delta n = \frac{1}{2}(n_1 - n_2)$. For the unstable subharmonic modes shown in figure 3 we have

$$8\Delta n = 1, 2, \ldots, 6. \tag{2.11}$$

Clearly the 'Doppler shift' is just equal to $c\Delta n$. From this it follows that in general the frequency of modulation in the stationary frame of reference is related to the basic frequency $f_d = 2\pi\sigma_d$ by

$$f_e/f_d = \Delta n - \alpha_n/\sigma_d. \tag{2.12}$$

$8\Delta n$	Δn	ak	α_n/σ_0	σ_d/σ_0	α_n/σ_d	f_e/f_d
1·0	0·125	0·072	0·062	1·003	0·062	0·063
2·0	0·250	0·155	0·118	1·012	0·117	0·133
3·0	0·375	0·246	0·163	1·031	0·158	0·217
4·0	0·500	0·309	0·198	1·049	0·189	0·311
5·0	0·625	0·351	0·221	1·063	0·209	0·416
6·0	0·700	0·382	0·231	1·075	0·215	0·535

TABLE 1. Calculation of the modulation frequency f_e induced by the most unstable mode

3. The most unstable mode

To determine the frequency of the *most unstable* mode at any given wave steepness ak we may proceed as follows.

In figure 3, we have drawn with a broken line the envelope of the solid curves. No solid curve lies above this envelope. Therefore at a point of contact of the envelope with any particular curve the corresponding mode has the fastest rate of growth of any mode at that particular value of the steepness ak. To determine the corresponding modulation frequency α_l we now go to figure 2 and read off the value of Re (σ) corresponding to that particular mode at the same value of ak.

The resulting values of α_l/σ_0 for each of the unstable modes in figure 3 are shown in table 1. Here σ_0 represents the radian frequency of infinitesimal waves ($ak \to 0$). To obtain the ratio α_l/σ_d we must divide by the relative speed of waves of finite amplitude, which may be calculated, for example, from figure 1 of Longuet-Higgins (1975). Finally f_e/f_d is found from equation (2.12).

The figures in the final column have been used to draw the curve in figure 5, showing σ_e/σ_d against ak. Also shown are the tangent at the origin, representing equation (1.2), and a vertical line marking the amplitude of the most unstable wave which, from figure 3, is found to occur at a steepness $ak = 0{\cdot}32$ approximately. It can be seen that for the lower part of the range of ak the corrected curve lies somewhat below the sloping line representing the Benjamin–Feir theory, but that it crosses this line at about $ak = 0{\cdot}30$ and thereafter lies above the line.

In fact since the steepest waves in an irregular sea will exceed the average steepness by a factor of 1·5 or more (Longuet-Higgins 1952) we hardly expect to find mean values of ak much above 0·3 and for those the corrected curve always lies below the Benjamin–Feir tangent.

4. Comparison with observation

In figure 6 the theoretical curve representing f_e/f_d has been inserted in figure 13 of Lake & Yuen (1978), which includes the data derived from a wind-wave channel at rather short fetches. It will be seen that the new curve agrees with the data better than the original theory, in so far as it goes more nearly through the centre of the cloud of observations.

However, it must be said, first, that the measurement of the envelope frequency f_e from a typical record may be a somewhat subjective procedure and, second, that there is no obvious reason to prefer the *mean* wave amplitude $\overline{ak}$ as a parameter of the wave

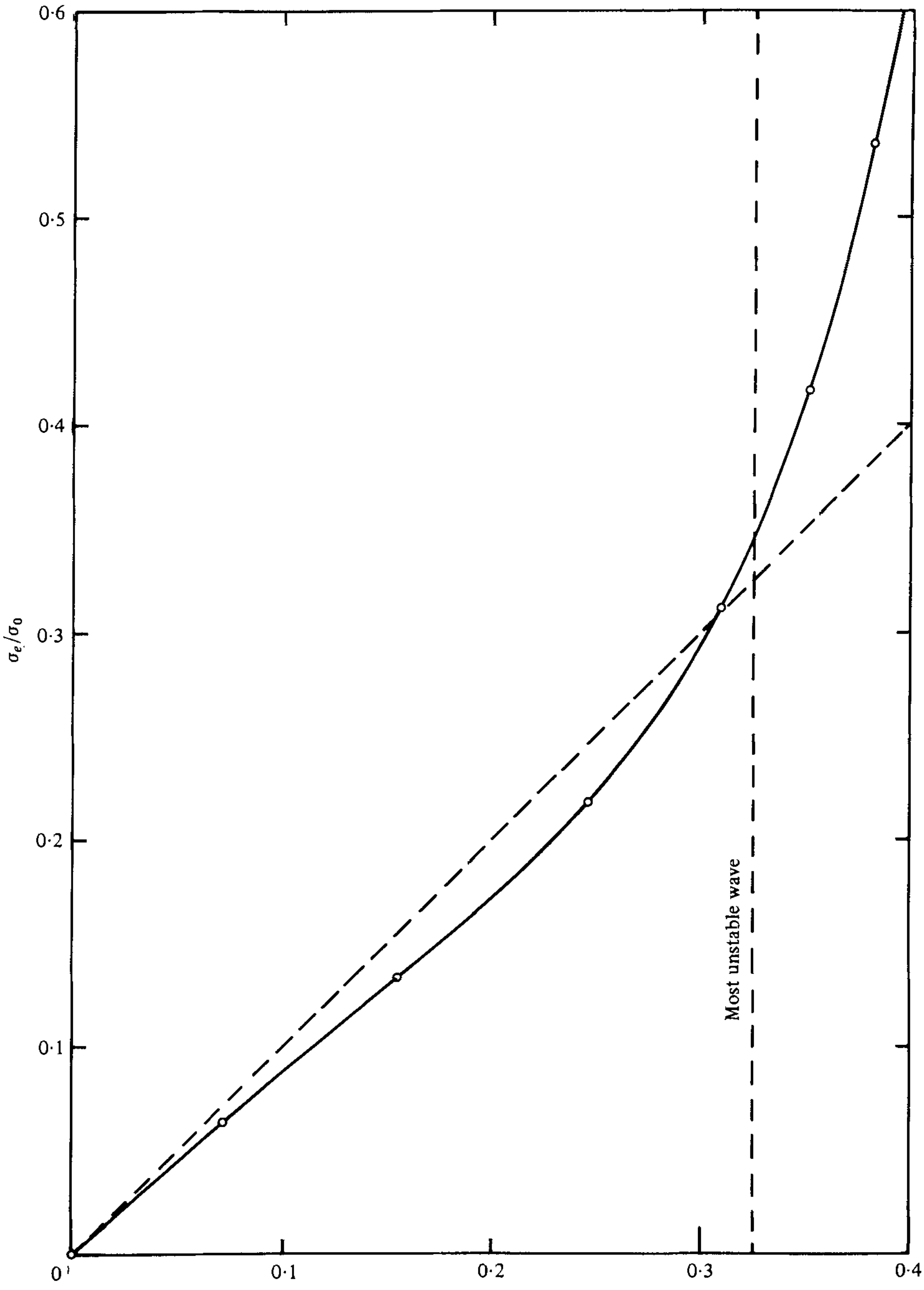

FIGURE 5. The frequency of modulation of the wave envelope in a wave of initial amplitude a perturbed by the fastest-growing instability.

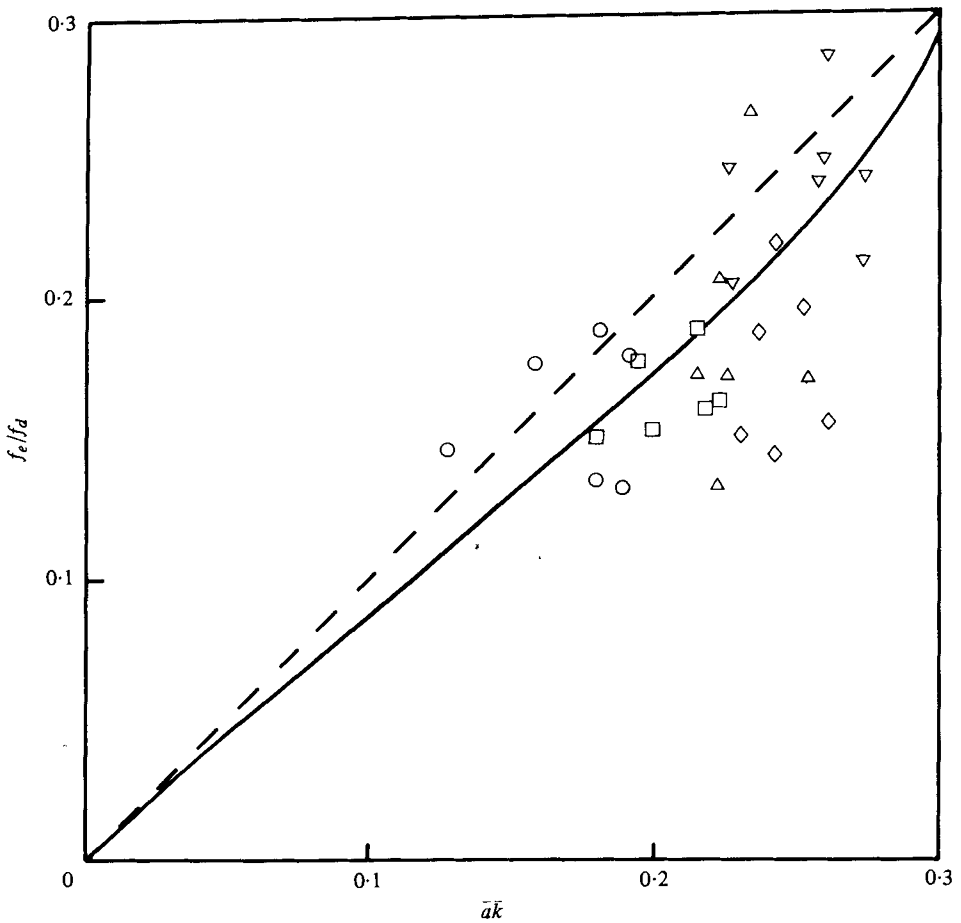

FIGURE 6. Ratio of the modulation frequency to the dominant wave frequency, accordng to the laboratory wind-wave data of Lake and Yuen (1978), compared to the most unstable modulation frequency. Broken line represents Benjamin & Feir's (1967) theory. Solid curve represents present calculations. ○, $U_W = 15 \text{ ft s}^{-1}$; □, $U_W = 20 \text{ ft s}^{-1}$; △, $U_W = 25 \text{ ft s}^{-1}$; ◇, $U_W = 30 \text{ ft s}^{-1}$; ▽, $U_W = 35 \text{ ft s}^{-1}$.

field rather than say the root-mean-square wave amplitude, which would be somewhat greater.

Nevertheless one can state, on the evidence of figure 6, that in the steepest waves encountered in these experiments the frequency of modulation was about 0·2 times the wave frequency, implying that in a very steep sea every fifth wave, on average, is the highest. And, further, that this is supported by our calculations.

It must be emphasized that this would be the wave grouping as seen by an observer or wave recorder at the *fixed point* $x = 0$, measuring the elevation as a function of the time. When seen in space, the wavenumber of the envelope would appear to be simply

$$\Delta n = \tfrac{1}{2}(n_1 - n_2). \tag{4.1}$$

For waves of low amplitude, and fairly long groups we have in deep water

$$\alpha_n = \Delta\sigma \doteqdot \tfrac{1}{2}c\,\Delta n, \tag{4.2}$$

since the group velocity equals half the phase velocity. Hence the wave groups appear twice as long in time as they do in space.

The lower the waves, according to these ideas, the longer the groups of waves become, until the width of the spectrum ceases to be governed by the local nonlinear dynamics and is determined instead by other factors depending on the synoptic wind pattern and the path of wave propagation.

For field measurements of ocean waves, the wave *age*, as given by the ratio c/U where U is wind-speed, is generally much greater than in the laboratory experiments reported by Lake & Yuen (1978). It appears that typically c/U lies between 0·5 and 1·5, and hence that ak, according to Sverdrup & Munk (1947) lies between 0·3 and 0·05, but with the bulk of the observations corresponding to $c/U \simeq 1{\cdot}0$, $ak \simeq 0{\cdot}10$. It will be interesting to see to what extent the observed values of f_e/f_d lie near the theoretical curve of figure 6.

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